

Gravitational lensing in fourth order gravity

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Gravitational lensing is investigated in the weak field limit of fourth order gravity in which the Lagrangian of the gravitational field is modified by replacing the Ricci scalar curvature R with an analytical expression $f(R)$. Considering the case of a pointlike lens, we study the behaviour of the deflection angle in the case of power-law Lagrangians, i.e. with $f(R) \propto R^n$. In order to investigate possible detectable signatures, the position of the Einstein ring and the solutions of the lens equation are evaluated considering the change with respect to the standard case. Effects on the amplification of the images and the Paczynski curve in microlensing experiments are also estimated.

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I. INTRODUCTION

The last decade observational results, including Type Ia Supernovae [1], cosmic microwave background anisotropy spectrum [2, 3] and large scale structure [4], have radically changed our view of the universe leading to revise the old cosmological standard model. On the contrary, the nowadays standard scenario, referred to as the *concordance model*, although still spatially flat, assumes that the visible baryonic matter constitutes only $\sim 4\%$ of the matter-energy content, while the most of the budget is in the form of *dark* components. This dark sector is assumed to be constituted for one third by *dark matter*, characterized by ordinary thermodynamical properties and mainly clustered in galaxies and clusters of galaxies, and for two third by a negative pressure fluid dubbed *dark energy*, which drives the large scale cosmic acceleration and does not cluster at large and small scales hence not influencing galactic dynamics.

Investigating the nature and the properties of the *dark side of the universe* represents the most daunting and yet fascinating challenge of modern cosmology so that it is not surprising that there are a lot of theoretical proposals on the ground. Nevertheless, both in the case of the old problem of dark matter and the more recent issue of dark energy, few approaches seem to furnish a quite natural and coherent framework within which to explain the observational results. Still more rare are models which provide a unified view of both dark matter and dark energy as a two separate manifestations of a single scale-dependent gravitational phenomenon.

From this point of view, higher order theories of gravity (both in the metric [5, 6, 7, 8] and the Palatini [9, 10, 11] formulations) represent an interesting approach able to fruitfully cope with both dark matter and dark energy

problems¹. On one hand, it has been widely demonstrated that such theories can agree with the cosmological observations of the Hubble flow [13, 14] and the large scale structure evolution [15]. On the other hand, in the weak field limit, the gravitational potential turns out to be modified [16, 17, 18, 19, 20] in such a way that interesting consequences on galactic dynamics may be achieved without violating, at the same time, the constraints on the PPN parameters [21] from Solar System tests.

It is worth remembering that one of the first experimental confirmations of Einsteinian general relativity was the deflection of light observed during the Solar eclipse of 1919. Since then, gravitational lensing (i.e., the deflection of light rays crossing the gravitational field of a compact object referred to as *lens*) has become one of the most astonishing successes of general relativity and it represents nowadays a powerful tool able to put constraints on different scales, from stars to galaxies and cluster of galaxies, to the large scale structure of the universe and cosmological parameters (see [22] and [23] for comprehensive textbooks).

Being intimately related to the underlying theory of gravity in its Einstein formulation, it is quite obvious that modifying the Lagrangian of the gravitational field also affects the theory of gravitational lensing. It is therefore mandatory to investigate how gravitational lensing works in the framework of higher order theories of gravity. On the one hand, one has to verify that the phenomenology of gravitational lensing is preserved in order to not contradict those observational results that do agree with the predictions of the *standard*² theory of lensing. On the other hand, it is worth exploring whether deviations

¹ An alternative yet similar approach is the one based on scalar-tensor theories of gravity (see, e.g., [12] and references therein).

² In the remaining of the paper, we will refer to the theory of gravitational lensing based on Einstein general relativity as the *standard* or *Einsteinian* theory, while as *corrected* theory we mean the one based on fourth order gravity.

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from classical results of the main lensing quantities could be somewhat detected and work as clear signatures of a modified theory of gravity. As a first step towards such an ambitious task, we consider here power-law fourth order theories, i.e. we replace the Ricci scalar R in the gravity Lagrangian with the function $f(R) \propto R^n$, and investigate how this affects the gravitational lensing in the case of a pointlike lens.

The paper is organized as follows. After a short summary of $f(R)$ theories, we derive, in Sect. III, the general expression for the deflection angle of a pointlike lens in the weak field regime. In particular, we discuss why this result holds whatever is the theory of gravity which enters in determining the expression for the gravitational field. Sect. IV is devoted to evaluating the deflection angle and deriving the lens equation for the case of the $f(R)$ theory we are considering. Since our results refer to the case of a pointlike lens, we may compare and contrast them with the classical ones in the microlensing regime. This issue is extensively discussed in Sect. V where we study the deviations from the classical results regarding the position of the images, the Einstein angle, the total amplification and the Paczynski lightcurve, while, in Sect. VI, we summarize and conclude.

II. BASICS OF $f(R)$ THEORIES

Higher order theories of gravity, also referred to as $f(R)$ theories, represent the most natural generalization of the Einsteinian general relativity. To this aim, one considers the gravity action :

$$\mathcal{A} = \int d^4x \sqrt{-g} [f(R) + \mathcal{L}_m] \quad (1)$$

where $f(R)$ is a generic function of the Ricci scalar curvature R differentiable at least up to the second order and \mathcal{L}_m is the standard matter Lagrangian. The choice $f(R) = R + 2\Lambda$ gives the general relativity including the contribution of the cosmological constant Λ . Varying the action with respect to the metric components $g_{\mu\nu}$, one gets the generalized Einstein equations that can be more expressively recast as [5] :

$$G_{\mu\nu} = \frac{1}{f'(R)} \left\{ \frac{1}{2} g_{\mu\nu} [f(R) - Rf'(R)] + f'(R)_{;\mu\nu} - g_{\mu\nu} \square f'(R) \right\} + \frac{T_{\mu\nu}^{(m)}}{f'(R)} \quad (2)$$

where $G_{\mu\nu} = R_{\mu\nu} - (R/2)g_{\mu\nu}$ and $T_{\mu\nu}^{(m)}$ are the Einstein tensor and the standard matter stress-energy tensor respectively and the prime denotes derivative with respect to R . The two terms $f'(R)_{;\mu\nu}$ and $\square f'(R)$ imply fourth order derivatives of the metric $g_{\mu\nu}$ so that these models are also referred to as *fourth order gravity*. Starting from

Eq.(2) and adopting the Robertson-Walker metric, it is possible to show that the Friedmann equations may still be written in the usual form provided that an *effective curvature fluid* is added to the matter term with energy density ρ_{curv} and pressure $p_{curv} = w_{curv}\rho_{curv}$ depending on the choice of $f(R)$ as :

$$\rho_{curv} = \frac{1}{f'(R)} \left\{ \frac{1}{2} [f(R) - Rf'(R)] - 3H\dot{R}f''(R) \right\}, \quad (3)$$

$$w_{curv} = -1 + \frac{\ddot{R}f''(R) + \dot{R}[\dot{R}f'''(R) - Hf''(R)]}{[f(R) - Rf'(R)]/2 - 3H\dot{R}f''(R)}. \quad (4)$$

As a particular case, we consider power-law $f(R)$ theories, i.e. we set :

$$f(R) = f_0 R^n \quad (5)$$

with n the slope of the gravity Lagrangian ($n = 1$ being the Einstein theory) and f_0 a constant with the dimensions chosen in such a way to give $f(R)$ the right physical dimensions. It has been shown that such a choice leads to matter only models able to fit well the Hubble diagram of Type Ia Supernovae without the need of dark energy [13] and could also be reconciled with the constraints on the PPN parameters [21].

III. THE DEFLECTION ANGLE

Although generalizing the gravity Lagrangian has a deep impact on both the cosmology and the local dynamics, it is nevertheless easy to understand that the derivation of the lens equation is formally the same as that in standard general relativity. Indeed, the basic assumption in deriving the lens equation is that the gravitational field is weak and stationary. In this case, the spacetime metric reads :

$$ds^2 = \left(1 - \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 + \frac{2\Phi}{c^2}\right) \delta_{ij} dx^i dx^j \quad (6)$$

where Φ is the gravitational potential and, as usual, we have neglected the gravitomagnetic term³. Since light rays move along the geodesics of the metric (6), the lens equation may be simply derived by solving $ds^2 = 0$ and keeping only terms up to second order in v/c . It is worth stressing that such a derivation holds whatever is the

³ Gravitomagnetic effects originate from the mass current in the lens and, under some circumstances, could give rise to effects that could be detected with future interferometric satellites (see, e.g., [24] and references therein).

theory of gravity provided that one can still write Eq.(6) in the approximation of weak and stationary fields. As a fundamental consequence, the lensing deflection angle will be given by the same formal expression that holds in general relativity [22, 23]

$$\vec{\alpha} = \frac{2}{c^2} \int \vec{\nabla}_\perp \Phi dl \quad (7)$$

with :

$$\vec{\nabla}_\perp \equiv \vec{\nabla} - \hat{e}(\hat{e} \cdot \vec{\nabla}) \quad (8)$$

where \hat{e} is the spatial vector⁴ tangent to the direction of the light ray and $dl = \sqrt{\delta_{ij} dx^i dx^j}$ is the Euclidean line element. The integral in Eq.(7) should be performed along the light ray trajectory which is *a priori* unknown. However, for weak gravitational fields and small deflection angles, one may integrate along the unperturbed direction. In this case, we may set the position along the light ray as :

$$\vec{r} = \vec{\xi} + l\hat{e} \quad (9)$$

with $\vec{\xi}$ orthogonal to the light ray. Assuming the potential only depending on $r = |\vec{r}| = (\xi^2 + l^2)^{1/2}$ (as for a pointlike or a spherically symmetric lens), the deflection angle reduces to :

$$\vec{\alpha} = \frac{2\vec{\xi}}{c^2} \int_{-\infty}^{\infty} \left(\frac{1}{r} \frac{d\Phi}{dr} \right) dl \quad (10)$$

where we have assumed that the geometric optics approximation holds, the light rays are paraxial and propagate from infinite distance.

Eq.(10) is general and holds whatever is the underlying theory of gravity. This nice result does not mean that the deflection angle is the same as in the Einsteinian general relativity. Indeed, the link with the underlying gravity theory is represented by the gravitational potential Φ which is determined by solving the corresponding generalized Einstein equations for the weak field metric (6). As a result, Φ can be no more the standard Newtonian potential and deviations from the Keplerian scaling $1/r$ enter the game leading to interesting consequences. From the lensing point of view, this means that, although Eq.(10) still holds, the deflection angle differs from the one of general relativity in a way that critically depends on the expression chosen for the function $f(R)$ entering the gravity Lagrangian.

Before deriving explicitly the deflection angle in $f(R)$ theories, it is worth discussing an important point. The above considerations rely on the assumption that the weak field metric may be still be written as in Eq.(6) in the case of alternative theories of gravity. That is, one is implicitly assuming that the solution of the field equations in the low energy limit shares the same formal structure as the Schwarzschild solution which Eq.(6) reduces to when $\Phi(r)$ is the Newtonian $1/r$ potential. Actually, under the hypothesis of weak gravitational fields and slow motions (both well verified in the astrophysical phenomena of interest), by virtue of the Birkhoff - Jensen theorem, the weak field metric is :

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2 d\Omega^2 \quad (11)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ is the line element on the unit sphere. Inserting Eq.(11) into the Einstein field equations one obtains the classical Schwarzschild solution by setting $A(r) = 1/B(r) = 1 + 2\Phi(r)/c^2$ and using the limit $\Phi/c^2 \ll 1$ with $\Phi(r)$ the Newtonian potential. One could argue that this is no more the case when alternative theories of gravity are considered so that one should take $A(r) \neq 1/B(r)$ and then deriving the lens equation in the general PPN formalism as in [25]. However, considering higher order theories of gravity leads to higher order field equations and hence widens the set of solutions. It is therefore possible, *a priori*, to find out solutions of the low energy limit of the field equations under the assumption $A(r) = 1/B(r)$ so that the space-time metric indeed reads as in Eq.(6) and the deflection angle is given by Eq.(10) also for $f(R)$ theories.

IV. THE POINTLIKE LENS

Eq.(10) allows to evaluate the deflection angle provided that the source mass distribution and the theory of gravity have been assigned so that one may determine the gravitational potential. As a first step, we consider here the case of the pointlike lens. Note that, although being the simplest one, the pointlike lens is the standard approximation for stellar lenses in microlensing applications [22, 26]. Moreover, since in the weak field limit $\vec{\alpha}$ is an additive quantity, the deflection angle for an extended lens may be computed integrating the pointlike result weighted by the deflector surface mass distribution under the approximation of thin lens [22] (i.e., the mass distribution extends over a scale which is far smaller than the distances between observer, lens and source).

Given the symmetry of the problem, it is clear that we may deal with the magnitude of the deflection angle and of the other quantities of interest rather than with vectors. In the approximation of small deflection angles, simple geometrical considerations allow to write the lens equation as :

⁴ Here, quantities with an over - arrow are vectors, while the vector will be denoted by an over - hat.

$$\theta - \theta_s = \frac{D_{ls}}{D_s} \alpha \quad (12)$$

which gives the position θ in the lens plane of the images of the source situated at the position θ_s on the source plane⁵. Note that the lens and the source planes are defined as the planes orthogonal to the optical axis which is the line joining the observer and the centre of the lens. Here and in the following, D_l , D_s , D_{ls} are the angular diameter distances between observer-lens, observer-source and lens-source respectively.

In order to evaluate the deflection angle, we need an explicit expression for the gravitational potential Φ generated by a pointlike mass m . This has been derived in detail in [20] for the case of power-law $f(R)$ theories. Solving the low energy limit of the field equations (2) in vacuum ($T_{\mu\nu}^{(m)} = 0$) for $f(R)$ given by Eq.(5), one obtains for the gravitational potential:

$$\Phi(r) = -\frac{Gm}{2r} \left[1 + \left(\frac{r}{r_c} \right)^\beta \right] \quad (13)$$

with:

$$\beta = \frac{12n^2 - 7n - 1 - \sqrt{36n^4 + 12n^3 - 83n^2 + 50n + 1}}{6n^2 - 4n + 2} \quad (14)$$

For $n = 1$, $\beta = 0$ and the potential reduces to the Newtonian one as expected. While the slope β of the correction term is a universal quantity (since it depends on the exponent n entering the gravity Lagrangian), the scalelength r_c is related to one of the integration constant that have to be set to solve the fourth order differential equations of the theory. As such, r_c is expected to be related to the peculiarities of each system and can therefore take different values depending on its mass and typical scale.

Inserting Eq.(13) into (10) and using $\xi = D_l \theta$, after some algebra, we finally get for the deflection angle:

$$\alpha = \frac{2Gm}{c^2 r_c} \left(\frac{\xi}{r_c} \right)^{-1} \left[1 + \frac{\sqrt{\pi}(1-\beta)\Gamma(1-\beta/2)}{2\Gamma(3/2-\beta/2)} \left(\frac{\xi}{r_c} \right)^\beta \right] \quad (15)$$

which is defined⁶ only for $0 \leq \beta \leq 2$. In the following, however, we will only consider the narrowest range $0 \leq \beta \leq 1$ since, as discussed in [20], in this case, the modified potential offers the possibility to fit galactic rotation curves without dark matter. As a cross check, note that, for $n = 1$, it is $\beta = 0$ and:

$$\lim_{\beta \rightarrow 0} \alpha(\xi, \beta) = \alpha_{GR} = \frac{4Gm}{c^2 \xi}$$

so that the classical⁷ result of Einsteinian relativity is recovered. On the contrary, for $\beta = 1$, we get $\alpha = \alpha_{GR}/2$ so that the net effect of the correction term is to decrease the deflection angle with respect to the classical one.

As we have quoted above and explicitly shown in [20], the modified gravitational potential leads to an increase of galactic rotation curve acting as a sort of *effective* dark matter filling the gap between the Newtonian and the observed rotation curve. Given that the classical deflection angle is proportional to the mass of the system, one could expect that the modified potential leads to an increase of the deflection angle while we get the opposite result. However, there is an important difference explaining this counterintuitive behaviour. Indeed, the deflection angle depends on the integral of $\mathcal{I} = (1/r)d\Phi/dr$. Comparing the Newtonian potential with the modified one, we get:

$$\frac{\Phi(r)}{\Phi_{GR}(r)} = \frac{1}{2} \left[1 + \left(\frac{r}{r_c} \right)^\beta \right],$$

$$\frac{\mathcal{I}(r)}{\mathcal{I}_{GR}(r)} = \frac{1}{2} \left[1 + (1-\beta) \left(\frac{r}{r_c} \right)^\beta \right],$$

so that

$$\Phi(r) > \Phi_{GR}(r) \iff r > r_c,$$

$$\mathcal{I}(r) > \mathcal{I}_{GR}(r) \iff r > (1-\beta)^{-\beta} r_c.$$

In order an increase of the rotation curve (and hence a good fit without dark matter), the first condition should be met which is easy to realize on galactic scales given typical values of r_c . On the contrary, for a pointmass lens, since $\beta < 1$, the integral entering the deflection angle $\mathcal{I}(r)$ is actually always smaller than the corresponding classical one $\mathcal{I}_{GR}(r)$ since $(1-\beta)^{-\beta} r_c$ is very large. As a consequence, then, the corrected deflection angle turns out to be lower than the classical one.

Inserting Eq.(15) into Eq.(12), the lens equation may be conveniently written as:

$$\left[1 - \mathcal{N}(\beta, \vartheta_c) \left(\frac{\vartheta}{\vartheta_c} \right)^{\beta-2} \right] \vartheta^2 - \vartheta_s \vartheta - \frac{1}{2} = 0 \quad (16)$$

⁵ Both θ and θ_s are measured in angular units and could be redefined as $\theta = \xi/D_l$ and $\theta_s = \eta/D_s$ with ξ and η in linear units.

⁶ For β outside this range, the deflection angle may still be computed, but there is not an analytical expression.

⁷ Hereafter, we will denote with the subscript *GR* quantities evaluated for $n = 1$, while we use the subscript *FOG* for the correction due to the fourth order theory. With this notation, for instance, the total deflection angle is $\alpha = \alpha_{GR} + \alpha_{FOG}$. Moreover, we refer to the results obtained for $n = 1$ as *classical*.

with

$$\mathcal{N}(\beta, \vartheta_c) = \frac{\sqrt{\pi}(1-\beta)\Gamma(1-\beta/2)}{4\vartheta_c^2\Gamma(3/2-\beta/2)} \quad (17)$$

having defined $\vartheta = \theta/\theta_{E,GR}$, with $\theta_{E,GR}$ the Einstein angle in the general relativity case given by :

$$\theta_{E,GR} = \sqrt{\frac{4GmD_{ls}}{c^2D_lD_s}}. \quad (18)$$

Note that, for $n = 1$ ($\beta = 0$), the corrected lens equation (16) reduces to the classical one :

$$\vartheta^2 - \vartheta_s\vartheta - 1 = 0$$

thus ensuring the self consistency of the theory. As a general remark, the modified lens equation differs from the classical one because of the second term in the square brackets which furnishes a correction depending on both β and r_c (through $\vartheta_c = r_c/D_l\theta_{E,GR}$). Actually, the magnitude of the correction strongly depends not only on the scaling radius r_c , but also on the lens mass and the distances (D_l, D_s, D_{ls}) since they all enter ϑ_c .

V. MICROLENSING OBSERVABLES

The pointlike lens equation (16) differs from the standard general relativistic one for the second term in the square parenthesis. Should this term be negligible, all the usual results of gravitational lensing are recovered. It is therefore interesting to investigate in detail how the corrective term affects the estimate of observable related quantities since, should they come out to be detectable, they could represent a signature of R^n gravity.

Since we are considering the pointlike lens, a typical lensing system is represented by a compact object (both visible or not) in the Galaxy halo acting as a lens on the light rays coming from a stellar source in an external galaxy, like the Magellanic Clouds (LMC and SMC) or Andromeda. It is easy to show that, in such a configuration, the standard Einstein angle $\theta_{E,GR}$ and the images angular separation are of the order of few $\times 10^{-5}$ arcsec so that we are in the regime known as *microlensing* [26].

In the following, we will consider different observable quantities in microlensing applications in order to see whether it is possible to detect deviations from the standard theory of gravitational lensing. As a fiducial system, we take a stellar lens in the galactic halo (with $m = 1 M_\odot$ and $D_l = 20$ kpc) and a source star in the LMC (with $D_s = 50$ kpc). The main results may be easily scaled to other values of (m, D_l, D_s) by the corresponding Newtonian Einstein angle.

The dependence on the theory is parametrized by the two parameters (n, r_c) entering the modified gravitational potential (13). Actually, we prefer to use β rather

than n as first parameter since its value is limited in the narrow range $(0, 1)$ with $\beta = 0$ corresponding to the General Relativity case $n = 1$. In order to estimate a reasonable range for r_c , we proceed as follows. Let us rewrite the potential as :

$$\Phi = -\frac{Gm}{2r_c}\eta^{-1}(1+\eta^\beta) = -\frac{\Phi_1}{2}\eta^{-1}(1+\eta^\beta)$$

with $\eta = r/r_c$ and $\Phi_1 = Gm/r_c$. Since Φ_1 has the dimension of a squared velocity, we introduce a further parameter $v_1 = \Phi_1^{1/2}$ so that the potential conveniently reads

$$\Phi = -\frac{c^2}{2}\left(\frac{v_1}{c}\right)^2\eta^{-1}(1+\eta^\beta).$$

Comparing the two expressions for the potential, we get :

$$r_c = \frac{Gm}{c^2\tilde{v}_1^2} \quad (19)$$

with $\tilde{v}_1 = v_1/c$. Rather than r_c , we will use in the following $\log \tilde{v}_1$ as parameter since its range is easier to evaluate. Let us consider, for instance, the Sun as the field source and a planet as a test particle. Since the Earth velocity is $\simeq 30$ km/s, we can indeed take $(-6, -4)$ as a reasonable range for $\log \tilde{v}_1$. As a final remark, note that Eq.(19) clearly shows why r_c depends on both the mass and the scale of the system under examination.

A. The Einstein angle and the position of the images

As a first application, let us consider the solution of the lens equation starting from the case of perfect alignment among observer, lens and source. Setting $\vartheta_s = 0$, Eq.(16) reduces to :

$$\left[1 - \mathcal{N}(\beta, \vartheta_c)\left(\frac{\vartheta}{\vartheta_c}\right)^{\beta-2}\right]\vartheta^2 = \frac{1}{2} \quad (20)$$

which, for $\beta = 0$ ($n = 1$) reduces to :

$$\vartheta^2 = 1 \iff \theta = \theta_{E,GR}$$

i.e. the image is the well known Einstein ring with angular radius equal to the Einstein angle defined in Eq.(18). In the general case, Eq.(20) must be solved numerically for given values of the model parameters $(\beta, \log \tilde{v}_1)$. As a general result, the Einstein ring still forms, but the corrected Einstein angle θ_E deviates from the classical one, i.e. $\vartheta_E \neq 1$. It is therefore interesting to study the magnitude of such deviations. To this aim, we define :

$$\varepsilon_E = \frac{\theta_E - \theta_{E,GR}}{\theta_{E,GR}} = \vartheta_E - 1$$

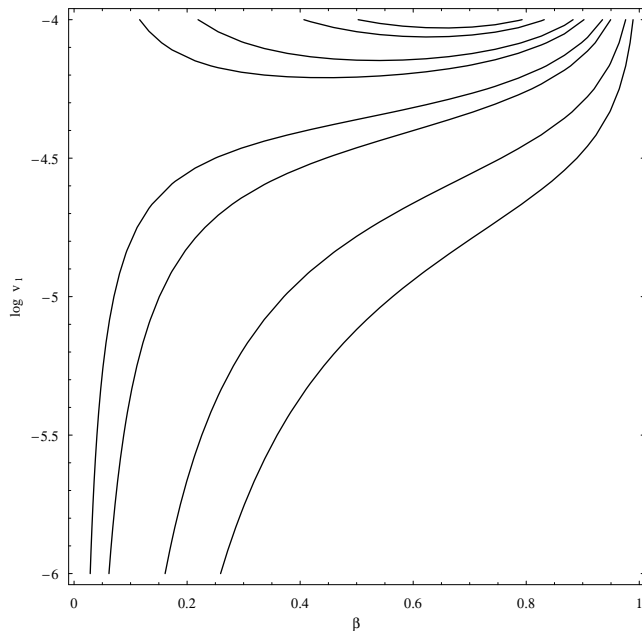


FIG. 1: Contours of equal ε_E in the plane $(\beta, \log \tilde{v}_1)$ for $\varepsilon_E = (-0.25, -0.20, -0.10, -0.05, 0.05, 0.10, 0.20, 0.25)$ from the bottom curve to the top one.

which quantifies the relative deviation with respect to the classical result. Contours of equal ε_E in the plane $(\beta, \log \tilde{v}_1)$ are shown in Fig. 1.

Some interesting remarks may be drawn from this plot. As a general result, ε_E could take both positive and negative values, i.e. R^n gravity may increase or decrease the Einstein angle with respect to the classical result. Actually, over almost the full parameter space, ε_E is negative and, in particular, taking $(\beta, \log \tilde{v}_1) \simeq (0.58, -5)$ as fiducial values⁸, we find $\varepsilon_E \simeq -0.25$ which leaves open the possibility to detect the corrections due to R^n gravity.

Before discussing this issue, it is worth examining how ε_E depends on the model parameters. For fixed values of β , the relative deviation is clearly an increasing function of $\log \tilde{v}_1$, a result that can be easily explained qualitatively. Indeed, the larger is $\log \tilde{v}_1$, the smaller is r_c and hence the earlier the correction term into the modified potential starts contributing thus explaining why the Einstein angle deviates more and more from the classical one as $\log \tilde{v}_1$ increase. A similar discussion may be done to explain why ε_E increases (in absolute value) with β for fixed $\log \tilde{v}_1$. To this aim, let us remember that the deflection angle α changes from α_{GR} for $\beta = 0$ to $\alpha_{GR}/2$ for $\beta = 1$ so that the lensing *strength* decreases with β . As a consequence, the Einstein angle becomes smaller as β gets higher and hence ε_E takes more negative values.

Modifying the Einstein angle has a deep impact on the microlensing phenomenology. As a first interesting application, let us remember that the duration of a microlensing event is estimated as $t_E = R_E/v_\perp$ with $R_E = D_l\theta_E$ the Einstein radius and v_\perp the transverse velocity of the lens with respect to the line of sight. Supposing that the distances (D_l, D_s) and the transverse velocity v_\perp were known (for instance, from deviations with respect to the standard lightcurve), a measurement of the Einstein time $t_{E,obs}$ translates into an estimate of the mass m of the lens. Denoting with μ the true mass (in units of $1 M_\odot$) and with μ_{GR} the one estimated from using incorrectly the classical expression for the Einstein angle, it is respectively:

$$t_{E,obs} = \mu^{1/2} t_E(m = 1 M_\odot) = \mu_{GR}^{1/2} t_{E,GR}(m = 1 M_\odot)$$

so that we get:

$$\mu = \frac{\mu_{GR}}{\vartheta_E^2(m = 1 M_\odot)}. \quad (21)$$

As can be inferred from Fig. 1, $\vartheta_E^2(m = 1 M_\odot) < 1$ over almost the full parameter space $(\beta, \log \tilde{v}_1)$ so that the true lens mass is higher than the one estimated by the classical method. However, in most of the applications, one may statistically infer only the lens transverse velocity, but not its distance so that what is indeed underestimated by the classical method is the quantity $\mu x(1-x)$ with $x = D_l/D_s$. As such, therefore, taking into account the correction to the Einstein angle may lead to an increase of either the estimated lens mass or of the distance ratio x . Disentangling these two effects is quite difficult and need for non standard microlensing events. A possible way out of this problem could be resorting to non standard microlensing events, but, in such a case, the duration of the event may differ from t_E so that the above qualitative argument does not apply anymore.

A similar discussion can be made for the optical depth τ that measures the probability of finding a lens in the microlensing tube, a cylinder with main axis along the line of sight and with radius equal to the Einstein radius. Since the corrected Einstein radius is smaller than the classical one, the microlensing tube is narrower and hence the optical depth, for a given spatial distribution of the lenses, is lower than the classical one.

As a second application, let us study how the images position change. In the Einsteinian case, the lens equation may be solved analytically and one gets two images with positions given by:

$$\vartheta_{\pm,GR} = \frac{1}{2} \left(\vartheta_s \pm \sqrt{\vartheta_s^2 + 4} \right). \quad (22)$$

In the current case, the lens equation (16) must be solved numerically for given values of the source position ϑ_s and the $f(R)$ parameters $(\beta, \log \tilde{v}_1)$. We still get two images on opposite sides of the lens with one image lying

⁸ Indeed, fitting the LSB rotation curves, one gets $\beta = 0.58 \pm 0.15$ [20], while $\log \tilde{v}_1 \simeq -5$ is suggested by considering a Jovian-like planet as test particle in the potential of a Sun-like lens.

inside and the other one outside the Einstein ring. The geometric configuration is therefore the same as in the standard case, but the positions are slightly changed. To quantify this effect, we have studied the contours of equal ε_{\pm} in the plane $(\beta, \log \tilde{v}_1)$ with ε_{\pm} defined in a similar way as ε_E . Not surprisingly, we get the same results as for the Einstein angle, i.e. the contours are parallel to those of ε_E but shifted by an amount depending on ϑ_s . This is easily explained noting that the lens equation is quite similar to the one for the Einstein angle, while the term depending on the R^n gravity parameters is the same.

B. Images amplification and Paczynski curve

Gravitational lenses work also as a sort of *natural telescopes* amplifying the luminosity of the source in such a way to make visible objects that are otherwise too faint to be detected. As a general rule, the amplification is given by the inverse of the Jacobian of lens mapping. In the case of a lens with cylindrical symmetry (and hence also for the pointlike lens), this reduces to [22]:

$$\mathcal{A} = \left| \frac{\vartheta_s}{\vartheta} \frac{d\vartheta_s}{d\vartheta} \right|^{-1}. \quad (23)$$

Starting from Eq.(16), after some algebra, one gets:

$$\mathcal{A} = \left| 1 - a^2 - [(2 - \beta)a + (1 - \beta)b + \beta]b \right|^{-1} \quad (24)$$

with:

$$a = (2\vartheta^2)^{-1}, \quad (25)$$

$$b = \mathcal{N}(\beta, \vartheta_c) (\vartheta/\vartheta_c)^{\beta-2}. \quad (26)$$

Note that we have considered the unsigned amplification since we are not interested to the parity of the images, but only to the luminosity variation. As a check of the consistency of the theory, it is easy to see that:

$$\lim_{\beta \rightarrow 0} \mathcal{A}(\vartheta, \beta) = \mathcal{A}_{GR} = \left| 1 - \frac{1}{\vartheta^4} \right|^{-1} \quad (27)$$

so that the General Relativity result is recovered for $n = 1$. At the other extreme, we get:

$$\lim_{\beta \rightarrow 1} \mathcal{A}(\vartheta, \beta) = \left| 1 - \frac{1}{4\vartheta^4} \right|^{-1} \quad (28)$$

so that the amplification is a increasing function of the slope parameter β . This is in contrast with the results obtained for the Einstein angle and the position of the

images. This result is not unexpected since the amplification is defined in terms of the inverse of the Jacobian mapping and therefore increases when this latter decreases. Since $\det J$ is a decreasing function of β , the behaviour of \mathcal{A} is obvious.

As a first important check, we have verified that the corrections to the amplification do not change the structure of the critical curves defined as the loci in the lens plane where the amplification gets formally infinite. Indeed, using Eq.(20), we see that $b(\vartheta_E) = 1 - a(\vartheta_E)$ that, inserted into Eq.(24), gives $\det J(\vartheta_E) = 0$ and hence $\mathcal{A}(\vartheta_E) = \infty$. Therefore, the critical curve is still the Einstein angle as in the classical case and the only caustic (which is obtained projecting the critical curve on the source plane) is $\vartheta_s = 0$ as usual.

Since in a microlensing event the two images are merged together (given the tiny angular separation of the order twice the Einstein angle), what we observe is the total luminosity which is obtained by multiplying the source luminosity by the sum of the amplification of the two images. In the Einsteinian case, inserting Eqs.(22) into the right hand side of Eq.(27), one gets for the total amplification:

$$\mathcal{A}_{tot,GR} = \mathcal{A}_+ + \mathcal{A}_- = \left| \frac{\vartheta_s^2 + 2}{\vartheta_s \sqrt{\vartheta_s^2 + 4}} \right|. \quad (29)$$

A similar relation for the corrected total amplification cannot be written since we do not have an analytical expression for the images position as a function of the source position. Actually, since the intrinsic source flux is unknown, we are unable to measure \mathcal{A}_{tot} so that detecting deviations from the standard theory is not possible at all. Nevertheless, a different strategy could be implemented. Assuming that the lens crosses the line of sight moving with a constant velocity, the source position as function of time t could be written as [27]:

$$\vartheta_s = \sqrt{\vartheta_0^2 + \left(\frac{t - t_0}{t_E} \right)^2} \quad (30)$$

with $\vartheta_0 = \vartheta_s(t = t_0)$, t_0 the time of closest approach to the lens to the line of sight and t_E the Einstein time defined above. As the time goes on, the source position changes according to Eq.(30) and hence the images position varies. As a consequence, also the total amplification \mathcal{A}_{tot} becomes a function of time. In the Einsteinian case, inserting Eq.(30) into Eq.(29), one gets the well known Paczynski lightcurve [27] having the peculiar signatures of uniqueness and symmetry of the bump and achromaticity. Using the numerical solutions of Eq.(16), we can work out a corrected Paczynski lightcurve for the case of power-law $f(R)$ theories. Some examples are shown in Fig. 2 where, without loss of generality, we have set $t_0 = 0$ and $t_E = 10$ d.

As it is apparent from the figure, both the sign and the magnitude of the deviations from the standard lightcurve

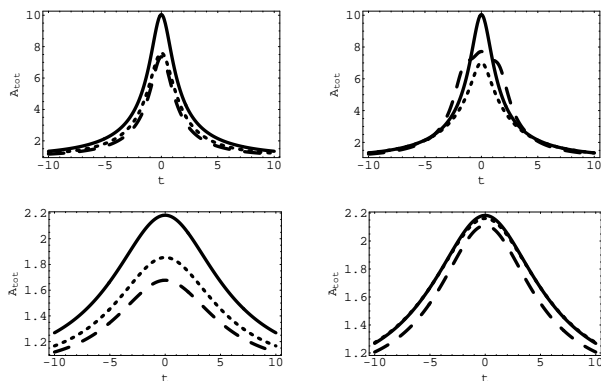


FIG. 2: Paczynski lightcurve for Einsteinian gravitational lensing (solid line) and the R^n gravity with $\beta = 0.25$ (short dashed) and $\beta = 0.75$ (long dashed). Upper (lower) panels refer to $\vartheta_0 = 0.1$ (0.5), while left (right) panels are for $\log \tilde{v}_1 = -5.5$ (-4.5).

depend in a complicated way on both the theory parameters ($\beta, \log \tilde{v}_1$) and the minimum impact parameter ϑ_0 . Although a general rule cannot be extracted from the plots, it is interesting to note, however, that, for some combination of the parameters, the microlensing lightcurve may be also distorted with respect to the Paczynski one. Nevertheless, the typical signature of microlensing events (symmetry, uniqueness of the bump and acromaticity) are preserved.

To estimate quantitatively the deviations from the Paczynski lightcurve, we plot in Fig. 3 the quantity $\Delta A_{tot} - 1$ as function of the time t for the same values of (t_0, t_E) . Considering, for instance, solid lines in the right panels, it is clear that the deviations are of order 10% which leaves open the possibility for a detection in the highest magnification events where the measurement errors on the lightcurve points are sufficiently low. A more detailed investigation is, however, needed to understand whether such deviations may be mimicked by other astrophysical effects. Indeed, we have here considered the case of a point mass acting as a lens on a point source. Should the lens (or the source) be a double system, this treatment breaks down and the curve may be distorted in a predictable way. Comparing these effects with those introduced by R^n gravity is outside our aim here, but it is a task worth to be addressed in detail to understand whether there are detectable signature of power-law $f(R)$ theories in the events lightcurves.

VI. CONCLUSIONS

Fourth order gravity has been recently proposed as viable alternatives to quintessence scalar fields and exotic fluids to solve the problem of cosmic speed up. In particular, power-law $f(R)$ theories have been shown to successfully fit the SNeIa data without violating the constraints

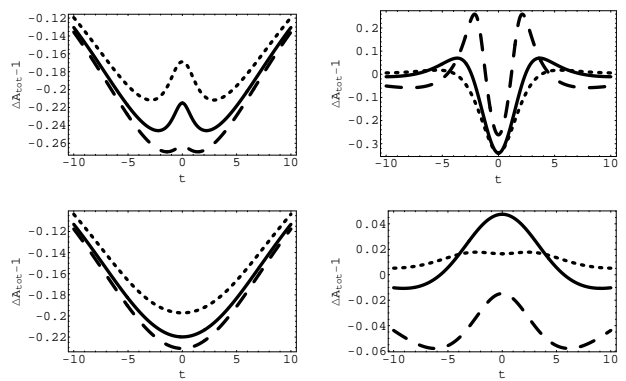


FIG. 3: Relative deviation with respect to the Paczynski lightcurve of the corrected microlensing lightcurve for R^n gravity with $\beta = 0.43$ (short dashed), $\beta = 0.58$ (solid) and $\beta = 0.73$ (long dashed). ϑ_0 and $\log \tilde{v}_1$ are set as in Fig. 3.

on the PPN parameters. Moreover, in the weak field limit, they give rise to a modified gravitational potential that makes it possible to fit the LSB rotation curves without the need of any dark matter halo.

Having successfully passed this impressive set of observational tests, the proposed modification to Einstein general relativity worths to be further investigated considering its effects on gravitational lensing. As a first step, we have derived an analytical expression for the deflection angle of a pointlike lens since this is the basic ingredient for a generalization to the case of extended systems (such as galaxies and clusters of galaxies). Because of the deviations of the gravitational potential from the Newtonian one, the deflection angle turns out to differ from the Einsteinian one because of an additive power-law term depending on the slope β and the scalelength r_c of the modified potential. For $0 \leq \beta \leq 1$, the corrected deflection angle is smaller than the classical one, but the amount of the corrections critically depends not only on β and r_c , but also on the lens mass m and the distances (D_l, D_s, D_{ls}) . As a consequence of the corrected deflection angle, the lens equation may no more be solved analytically and the resort to numerical techniques is needed. The number of images is still two with an angular separation of the order of the corrected Einstein angle.

To quantitatively investigate the impact of the corrections on observable quantities, we have considered the case of a solar mass object in the Galactic halo acting as a lens on a source star in the LMC so that we are in the typical conditions of microlensing regime. Denoting with ε_E the relative deviation of the corrected Einstein angle with respect to the classical one, we find out that both its sign and magnitude depend on the region in parameter plane $(\beta, \log \tilde{v}_1)$ considered, but significant deviations ($\varepsilon_E \sim -20\%$) may be obtained for the value of β ($\simeq 0.58$) suggested by the fit to the LSB galaxies rotation curves. A similar conclusion also holds for the deviations from the classical Paczynski lightcurve. Although a more de-

tailed analysis (also including non standard microlensing events and observational uncertainties) is needed, these preliminary results suggest the intriguing possibility to detect R^n gravity signatures through a careful examination of galactic microlensing. This may open the way to a completely new regime in which looking for corrections to the Einsteinian general relativity that may be complementary to the other cosmological probes.

It is worth stressing that, since significant deviations from the standard theory of microlensing are possible, the current estimates of the optical depth τ towards LMC (or any other target such as the galactic bulge, SMC or Andromeda) and of the mean lens mass inferred from the time duration distribution of microlensing events could be altered. In particular, the classical estimates of the optical depth (the mean lens mass) should be revised downward (upward) for fiducial values of the parameters $(\beta, \log \tilde{v}_1)$. Interestingly, microlensing surveys towards LMC have concluded that less than 20% of the putative dark halo is made out of compact baryonic objects (see, e.g., [29] and references therein) and there are models explaining the observed optical depth as a result of known stellar populations only belonging to our Galaxy [30] and/or to the LMC [31]. On the other hand, our modified gravity makes it possible to fit the Milky Way rotation curve without any dark matter halo. Indeed, one may speculate that self-lensing models should be preferred in the context of power-law $f(R)$ theories since one should have the possibility of explaining microlensing events without any dark matter halo (which is not needed in the $f(R)$ theory we are considering). As a test, one

could try to fit the Milky Way and LMC rotation curves using the modified gravitational potential and the mass distributions of the above quoted models. A successful result should be a serious evidence against dark matter halo in our Galaxy.

The pointlike lens is only the first and essential step towards the formulation of a theory of gravitational lensing in a power-law $f(R)$ theory. As a second step, we have to investigate the case of an extended lens such as a galaxy. This will offer us the possibility to get strong constraints on the theory parameters $(\beta, \log \tilde{v}_1)$ by fitting to the images configuration in multiply imaged quasars. This will also represent a critical test of the $f(R)$ theory since it is expected that β (and hence n) is the same for all the galaxy lenses considered. With more than 90 systems (see, e.g., the CASTLES survey website [32]), this test will be statistically meaningful and will be presented in a forthcoming paper.

As a final remark, we briefly comment upon the possibility of evaluating the corrected deflection angle for other classes of $f(R)$ theories. In principle, one should only insert the corresponding expression for the modified gravitational potential Φ in the general equation (10) for the deflection angle. Given the fourth order degree of the field equations, it is likely that the solution for Φ may still be approximated (at least, within a limited range) by Eq.(13) with $(\beta, \log \tilde{v}_1)$ related to the $f(R)$ parameters. It is therefore likely that the results discussed here for R^n gravity may be qualitatively extended to a more general class of higher order theories of gravity.

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- [1] A. G. Riess et al., AJ, 116, 1009, 1998; S. Perlmutter et al., ApJ, 517, 565, 1999; R.A. Knop et al., ApJ, 598, 102, 2003; J.L. Tonry et al., ApJ, 594, 1, 2003; B.J. Barris et al., ApJ, 602, 571, 2004; A.G. Riess et al., ApJ, 607, 665, 2004; A. Clocchiati et al., astro-ph/0510155, 2005; P. Astier et al., astro-ph/0510447, 2005
 - [2] P. de Bernardis et al., Nature, 404, 955, 2000; A. Balbi et al., ApJ, 545, L1, 2000; S. Hanany et al., ApJ, 545, L5, 2000; T.J. Pearson et al., ApJ, 591, 556, 2003; C.J. MacTavish et al., astro-ph/0507583, 2005
 - [3] C.L. Bennett et al., ApJS, 148, 1, 2003; D.N. Spergel et al., ApJS, 148, 175, 2003
 - [4] S. Dodelson et al., ApJ, 572, 140, 2002; W.J. Percival et al., MNRAS, 337, 1068, 2002; A.S. Szalay et al., ApJ, 591, 1, 2003; E. Hawkins et al., MNRAS, 346, 78, 2003; A.C. Pope et al., ApJ, 607, 655, 2004; S. Cole et al., MNRAS, 362, 505, 2005
 - [5] S. Capozziello, Int. J. Mod. Phys. D, 11, 483, 2002
 - [6] S. Capozziello, S. Carloni, A. Troisi, Recent Research Developments in Astronomy and Astrophysics, Research Signpost Publisher, astro-ph/0303041, 2003
 - [7] S. Carloni, P.K.S. Dunsby, S. Capozziello, A. Troisi, gr-qc/0410046, 2004; S. Capozziello, V.F. Cardone, A. Troisi, Phys. Rev. D, 71, 043503, 2005
 - [8] S. Nojiri, S.D. Odintsov, Phys. Lett. B, 576, 5, 2003; S. Nojiri, S.D. Odintsov, Mod. Phys. Lett. A, 19, 627, 2003; S. Nojiri, S.D. Odintsov, Phys. Rev. D, 68, 12352, 2003; S.M. Carroll, V. Duvvuri, M. Trodden, M. Turner, Phys. Rev. D, 70, 043528, 2004
 - [9] D.N. Vollick, Phys. Rev. 68, 063510, 2003; X.H. Meng, P. Wang, Class. Quant. Grav., 20, 4949, 2003; E.E. Flanagan, Class. Quant. Grav., 21, 417, 2004; X.H. Meng, P. Wang, Class. Quant. Grav., 21, 951, 2004; G.M. Kremer and D.S.M. Alves, Phys. Rev. D, 70, 023503, 2004
 - [10] S. Nojiri, S.D. Odintsov, Gen. Rel. Grav., 36, 1765, 2004; X.H. Meng, P. Wang, Phys. Lett. B, 584, 1, 2004
 - [11] G. Allemandi, A. Borowiec, M. Francaviglia, Phys. Rev. D, 70, 043524, 2004; G. Allemandi, A. Borowiec, M. Francaviglia, Phys. Rev. D, 70, 103503, 2004; G. Allemandi, A. Borowiec, M. Francaviglia, S.D. Odintsov, gr-qc/0504057, 2005
 - [12] V. Faraoni, *Cosmology in scalar-tensor gravity*, Kluwer Academic Publishers, 2004
 - [13] S. Capozziello, V.F. Cardone, S. Carloni, A. Troisi, Int. J. Mod. Phys. D, 12, 1969, 2003;
 - [14] S. Capozziello, V.F. Cardone, M. Francaviglia, astro-ph/0410135;
 - [15] M. Amarzguoui, O. Elgaroy, D.F. Mota, T. Multamaki, astro-ph/0510159; P. Zhang, astro-ph/0511218, 2005
 - [16] K.S. Stelle, Gen. Rel. Grav., 9, 353, 1978
 - [17] I. Quandt, H.J. Schmidt, Astron. Nachr., 312, 97, 1991
 - [18] P.D. Mannheim, ApJ, 419, 150, 1993

- [19] S. Capozziello, V.F. Cardone, S. Carloni, A. Troisi, Phys. Lett. A, 326, 292, 2004
- [20] S. Capozziello, V.F. Cardone, A. Troisi, astro-ph/0603522, 2006
- [21] G. Allemandi, M. Francaviglia, M.L. Ruggiero, A. Tartaglia, gr-qc/0506123, 2005; S. Capozziello, A. Troisi, astro-ph/0507545, 2005
- [22] P. Schneider, J. Ehlers, E.E. Falco, *Gravitational lenses*, Springer-Verlag, Berlin, 1992
- [23] A.O. Petters, H. Levine, J. Wambsganss, *Singularity theory and gravitational lensing*, Birkhäuser, Boston, 2001
- [24] S. Capozziello, G. Lambiase, G. Papini, G. Scarpetta, Phys. Lett. A, 254, 11, 1999; S. Capozziello, V. Re, Phys. Lett. A, 290, 115, 2001; M. Sereno, Phys. Lett. A, 305, 7, 2002; M. Sereno, V.F. Cardone, A&A, 396, 393, 2002; S. Capozziello, V.F. Cardone, V. Re, M. Sereno, MNRAS, 343, 360, 2003; M. Sereno, MNRAS, 357, 1205, 2005
- [25] C.R. Keeton, A.O. Petters, Phys. Rev. D, 72, 104006, 2005; C.R. Keeton, A.O. Petters, Phys. Rev. D, 73, 044024, 2006
- [26] S. Mollerach, E. Roulet, *Gravitational lensing and microlensing*, World Scientific Publisher, Singapore, 2002
- [27] B. Paczynski, ApJ, 506, 1, 1986
- [28] K. Griest, ApJ, 366, 412, 1991; A. de Rujula, Ph. Jetzer, E. Massó, MNRAS, 250, 348, 1991
- [29] Ph. Jetzer, A. Milsztajn, P. Tisserand, in *Gravitational lensing impact on cosmology*, IAU Symp. no. 225, Cambridge University Press, UK, 2005 ([arXiv: astro-ph/0409496](http://arxiv.org/abs/astro-ph/0409496))
- [30] D. Zaritsky, D.N.C. Lin, AJ, 114, 254, 1997; N.W. Evans, G. Gyuck, M.S. Turner, J.J. Binney, ApJ, 501, L45, 1998; H.S. Zhao, MNRAS, 294, 139, 1998
- [31] K. Sahu, Nature, 370, 275, 1994; X.P. Wu, ApJ, 435, 66, 1994; E.J. Kerins, N.W. Evans, ApJ, 517, 734, 1999; N.W. Evans, E.J. Kerins, ApJ, 529, 917, 2000; H.S. Zhao, N.W. Evans, ApJ, 545, L35, 2000
- [32] C.S. Kochanek, E.E. Falco, C. Impey, J. Lehar, B. McLeod, H.W. Rix, CASTLES webpage: <http://www.cfa.harvard.edu/castles>